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COEFFICIENT ESTIMATES FOR A CERTAIN SUBCLASS OF BI-UNIVALENT FUNCTIONS

ŞAHSENE ALTINKAYA - SIBEL YALÇIN

In the present investigation, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the function class $S_{\Sigma}(\lambda, h)$. The results presented in this paper improve or generalize the recent work of Magesh and Yamini.

1. Introduction and Definitions

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Further, by S we shall denote the class of all functions in A which are univalent in U .

The Koebe one-quarter theorem [9] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

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and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots.$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U .

Let Σ denote the class of bi-univalent functions defined in the unit disk U . For a brief history and interesting examples in the class Σ , (see [20]).

Lewin [14] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu [18] showed that $\max |a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$.

Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α ($0 < \alpha \leq 1$) respectively (see [18]). Thus, following Brannan and Taha [4], a function $f(z) \in A$ is said to be in the class $S_\Sigma^*(\alpha)$ of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$) if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, z \in U)$$

and

$$\left| \arg \left(\frac{wg'(w)}{g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, w \in U)$$

and is said to be in the class $K_\Sigma(\alpha)$ of strongly bi-convex functions of order α ($0 < \alpha < 1$) if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \left| \arg \left(1 + \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, z \in U)$$

and

$$\left| \arg \left(1 + \frac{wg'(w)}{g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, w \in U)$$

where g is the extension of f^{-1} to U . The classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α , corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For

each of the function classes $S_{\Sigma}^*(\alpha)$ and $K_{\Sigma}(\alpha)$, they found non-sharp estimates on the initial coefficients. In fact, the aforecited work of Srivastava et al. [20] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([1–3, 7, 8, 10, 13, 15–17, 19, 20, 22, 23]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([6, 11, 12]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, \dots\}$) is still an open problem.

In this paper, by using the method [21] different from that used by other authors, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for the subclasses of bi-univalent functions considered by Magesh and Yamini [16] and obtain a better estimate.

2. Coefficient Estimates

Definition 2.1. Let the functions $h, p : U \rightarrow \mathbb{C}$ be so constrained that

$$\min \{\Re(h(z)), \Re(p(z))\} > 0$$

and

$$h(0) = p(0) = 1.$$

Definition 2.2. A function $f \in \Sigma$ is said to be in the class $S_{\Sigma}(\lambda, h)$, $0 \leq \lambda \leq 1$, if the following conditions are satisfied:

$$\frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \in h(U) \quad (z \in U) \quad (2)$$

and

$$\frac{wg'(w) + (2\lambda^2 - \lambda)w^2g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} \in p(U) \quad (w \in U) \quad (3)$$

where $g(w) = f^{-1}(w)$.

Theorem 2.3. Let f given by (1) be in the class $S_{\Sigma}(\lambda, h)$. Then

$$|a_2| \leq \min \left\{ \sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2(1 + 3\lambda - 2\lambda^2)^2}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{4(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1)}} \right\} \quad (4)$$

and

$$|a_3| \leq \min \left\{ \frac{|h'(0)|^2 + |p'(0)|^2}{2(1+3\lambda-2\lambda^2)^2} + \frac{|h''(0)| + |p''(0)|}{8(1+2\lambda^2)}, \right. \\ \left. \frac{(12\lambda^4 - 28\lambda^3 + 19\lambda^2 + 2\lambda + 3)|h''(0)| + |12\lambda^4 - 28\lambda^3 + 11\lambda^2 + 2\lambda - 1||p''(0)|}{8(1+2\lambda^2)(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1)} \right\}. \quad (5)$$

Proof. Let $f \in S_\Sigma(\lambda, h)$, $0 \leq \lambda \leq 1$. It follows from (2) and (3) that

$$\frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} = h(z) \quad (6)$$

and

$$\frac{wg'(w) + (2\lambda^2 - \lambda)w^2g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} = p(w) \quad (7)$$

where $h(z)$ and $p(w)$ satisfy the conditions of Definiton 2.1. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_1z + h_2z^2 + \dots$$

and

$$p(w) = 1 + p_1w + p_2w^2 + \dots,$$

respectively. Since

$$\frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \\ = 1 + (1 + 3\lambda - 2\lambda^2)a_2z + \left[(12\lambda^4 - 28\lambda^3 + 11\lambda^2 + 2\lambda - 1)a_2^2 + (4\lambda^2 + 2)a_3 \right] z^2 + \dots \quad (8)$$

and

$$\frac{wg'(w) + (2\lambda^2 - \lambda)w^2g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} \\ = 1 - (1 + 3\lambda - 2\lambda^2)a_2w + \left[(12\lambda^4 - 28\lambda^3 + 19\lambda^2 + 2\lambda + 3)a_2^2 - (4\lambda^2 + 2)a_3 \right] w^2 + \dots, \quad (9)$$

it follows from (6), (7), (8) and (9) that

$$(1 + 3\lambda - 2\lambda^2)a_2 = h_1, \quad (10)$$

$$(12\lambda^4 - 28\lambda^3 + 11\lambda^2 + 2\lambda - 1)a_2^2 + (4\lambda^2 + 2)a_3 = h_2, \quad (11)$$

and

$$-(1 + 3\lambda - 2\lambda^2)a_2 = p_1, \quad (12)$$

$$(12\lambda^4 - 28\lambda^3 + 19\lambda^2 + 2\lambda + 3)a_2^2 - (4\lambda^2 + 2)a_3 = p_2. \quad (13)$$

From (10) and (12) we obtain

$$h_1 = -p_1,$$

and

$$2(1 + 3\lambda - 2\lambda^2)^2 a_2^2 = h_1^2 + p_1^2. \quad (14)$$

By adding (13) to (11), we find that

$$2(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1)a_2^2 = h_2 + p_2. \quad (15)$$

Therefore, we find from (14) and (15) that

$$|a_2|^2 \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2(1 + 3\lambda - 2\lambda^2)^2}.$$

and

$$|a_2|^2 \leq \frac{|h''(0)| + |p''(0)|}{4(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1)}.$$

So we get the desired estimate on the coefficient $|a_2|$ as claimed in (4).

Next, in order to find the bound on $|a_3|$, by subtracting (13) from (11), we obtain

$$2(2 + 4\lambda^2)a_3 - 2(2 + 4\lambda^2)a_2^2 = h_2 - p_2.$$

Then, in view of (14) and (15), it follows that

$$a_3 = \frac{h_1^2 + p_1^2}{2(1 + 3\lambda - 2\lambda^2)^2} + \frac{h_2 - p_2}{2(2 + 4\lambda^2)}$$

and

$$a_3 = \frac{h_2 + p_2}{2(12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1)} + \frac{h_2 - p_2}{2(2 + 4\lambda^2)}$$

as claimed. This completes the proof of Theorem 2.3. \square

Remark 2.4. We note that for $\lambda = \frac{1}{2}$, the class $S_\Sigma(\lambda, h)$ reduces to the class $H_\Sigma(h)$ studied by Srivastava et al. [21].

Corollary 2.5 (see [21]). *When $\lambda = \frac{1}{2}$ the results discussed in this article reduce to results in ([21], Corollary 3). If $f \in H_\Sigma(h)$ then*

$$|a_2| \leq \min \left\{ \sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{8}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{12}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{|h'(0)|^2 + |p'(0)|^2}{8} + \frac{|h''(0)| + |p''(0)|}{12}, \frac{|h''(0)|}{6} \right\}.$$

Corollary 2.6. *If*

$$\phi(z) = \left(\frac{1+z}{1-z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots \quad (0 < \alpha \leq 1),$$

then inequalities (4) and (5) become

$$|a_2| \leq \min \left\{ \frac{2\alpha}{1+3\lambda-2\lambda^2}, \sqrt{\frac{2}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}} \alpha \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{4\alpha^2}{(1+3\lambda-2\lambda^2)^2} + \frac{\alpha^2}{1+2\lambda^2}, \frac{2\alpha^2}{1+2\lambda^2} \right\}.$$

Remark 2.7. The estimates on the coefficients $|a_2|$ and $|a_3|$ of Corollary 2.6 are improvement of the estimates obtained in (Theorem 2.1, [16] and Corollary 2.3, [15]).

Remark 2.8. By setting $\lambda = 0$ in Corollary 2.6 we get the following consequence.

Corollary 2.9. *Let the functions $f(z)$ given by Taylor-Maclaurin series expansion (1) be in the bi-univalent function class $S_\Sigma^*(\alpha)$ ($0 < \alpha \leq 1$). Then*

$$|a_2| \leq \sqrt{2}\alpha$$

and

$$|a_3| \leq 2\alpha^2.$$

Corollary 2.10. *If*

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)z + 2(1-\alpha)^2 z^2 + \dots \quad (0 < \alpha \leq 1),$$

then inequalities (4) and (5) become

$$|a_2| \leq \min \left\{ \frac{2(1-\alpha)}{1+3\lambda-2\lambda^2}, \sqrt{\frac{2(1-\alpha)}{12\lambda^4-28\lambda^3+15\lambda^2+2\lambda+1}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{4(1-\alpha)^2}{(1+3\lambda-2\lambda^2)^2} + \frac{1-\alpha}{1+2\lambda^2}, \frac{(1-\alpha)}{1+2\lambda^2} \right\}.$$

Remark 2.11. The estimates on the coefficients $|a_2|$ and $|a_3|$ of Corollary 2.10 are improvement of the estimates obtained in (Theorem 3.1, [16] and Corollary 3.3, [15]).

Remark 2.12. By setting $\lambda = 0$ in Corollary 2.10 we get the following consequence.

Corollary 2.13 (see [21]). *Let the functions $f(z)$ given by Taylor-Maclaurin series expansion (1) be in the bi-univalent function class $S_{\Sigma}^*(\alpha)$ ($0 < \alpha \leq 1$). Then*

$$|a_2| \leq \sqrt{2(1-\alpha)}$$

and

$$|a_3| \leq 1 - \alpha.$$

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ŞAHSENE ALTINKAYA

Department of Mathematics, Faculty of Arts and Science

Uludag University

e-mail: sahsene@uludag.edu.tr

SIBEL YALÇIN

Department of Mathematics, Faculty of Arts and Science

Uludag University

e-mail: syalcin@uludag.edu.tr